MA/MSc - Mathematics (Final)
Assignments
COMPLEX ANALYSIS

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. (a) Let $G \subseteq \mathbb{C}$ be open and r : $[a,b] \to G$ be a rectifiable path. If $f : G \to \mathbb{C}$ is continuous then for each $\in > 0$, show that there is a polygonal path in G from a to b such that

   .

   (b) State and prove Liouvilles theorem and deduce fundamental theorem of algebra.

2. (a) Show that any Mobius transformation is a composition of translations, dialations and the inversions.

   (b) If G is a subset of the complex plane containing a closed disc (a,r) with $r > 0$ and then show that for any analytic function $f : G \to \mathbb{C}$

   $$f(z) = \frac{1}{2\pi i}\int_r \frac{f(\omega)}{(\omega - z)} d\omega \quad \text{for all } z \in B(a,r).$$

3. (a) State and prove the first version of Cauchy's integral formula.

   (b) If G is an open connected subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is any analytic function with zeros $a_1, a_2, \ldots, a_n$ (repeated according to multiplicity) in G and r is a closed rectifiable curve in G not passing though any of the zeros such that r $\approx$ 0, then show that

   .

4. (a) Write all possible Laurent series expansions for the function $f(z) = \dfrac{1}{z(z-1)(z-2)}$.

   (b) Evaluate .

5. (a) Show that $(C(G, \Omega), p)$ is a metric space.

   (b) If $|z| \leq 1$ and p 0, show that $|1 - E_p(z)| \leq |z|^{p+1}$.

6. (a) State and prove Montel's theorem.

   (b) Show that the set M(G) { } is a complete metric space.

7. (a) State and prove Runge's theorem.

   (b) be a path from a to b and $\{(f_t, D_t) / t \in [0,1]\}$ and $\{(g_t, B_t) / t \in [0,1]\}$ be analytic continuations along such that $[f_0]_a = [g_0]_a$. Then show that $[f_1]_b = [g_1]_b$.

8. (a) State and prove Monodromy theorem.

   (b) Define MVP and show that every harmonic function has MVP.

MA/MSc - Mathematics (Final)
Assignments

UNIVERSAL ALGEBRA

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. (a) Define the terms:

    (i) Partial order relation on the set *A*.

    (ii) Totally ordered set.

   (b) Prove that the following are equivalent in any lattice :

    (i) x (y z) = (x y) (x z).

    (ii) x (y z) = (x y) (x z).

   (c) Define the terms:

    (i) Heyting algebra.

    (ii) Ortho lattice.

   (d) Prove that if $\alpha : A \rightarrow B$ is an embedding, then α (*A*) is sub-universe of *B*.

   (e) Prove that *A* is directly indecomposable iff the only factor congruences on *A* are and .

   (f) Describe free semilattice.

   (g) Let *B* be a Boolean algebra, then prove that for any $a, b \in B$.

    (i) $a \quad b = 0$ and $a \quad b = 1$ imply $a = b'$.

    (ii) $(a \quad b)' = a' \quad b'$ and $(a \quad b)' = a' \quad b'$.

   (h) Let *B* be a Boolean algebra, then prove the following:

    (i) For, $I \quad B$, I is ideal iff $I' = \{a'/a \in I\}$ is filter.

    (ii) For, $F \quad B$, F is a filter iff $F' = \{a'/a \in F)$ is an ideal.

2. (a) Prove that two lattices L_1 and L_2 are isomorphic iff there is a bijection α from L_1 to L_2 such that both α and α^{-1} are order-preserving.

   (b) Prove that *L* is a non-modular lattice iff N_5 can be embedded into *L*.

3. (a) Prove that if *L* is a distributive lattice, then *I(L)* of all ideals of *L* is a distributive lattice.

   (b) Prove that *Eq(A)* of all equivalence relations on *A* is a complete lattice with respect to “ ”.

4. (a) Prove that is L is an algebraic lattice, then L $Sub(A)$, for some algebra A.

(b) Prove that ($Con\ A,$) is a complete sublattice of ($Eq(A),$), the lattice of equivalent relations on A.

5. (a) Let A be an algebra and suppose $Con A$, then prove that the following are equivalent:

(i)

(ii)

(iii)

(b) Let $\alpha : A \rightarrow B$ be a homomorphism onto B. Then prove that there is an isomorphism β from A / $ker(\alpha)$ to B defined by where is the natural homomorphism from A to A/ker(α).

6. (a) Prove that if , * is a pair of factor congruence on A then A A | x A| * under the map $\alpha(a) = (a/\ ,\ a/\ *)$.

(b) For an indexed family of maps $\alpha_i : A \rightarrow A_i$, $i \in I$, prove the following are equivalent:

(i) the maps α_i separate points.

(ii) α is injective.

(iii) $\bigcap_{i \in I}$ [illegible]

7. (a) Prove that every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebra (which one homomorphic images of A).

(b) Let K be a non-empty class of algebras of type F then prove that for some cardinal m, if |X| $\geq m$ we have $F_K(\ ) \in IP_S(K)$.

8. (a) Prove that every Boolean algebra is isomorphic to a subdirect power of 2, hence every Boolean algebra is isomorphic to a field of sets.

(b) Let (R, +, ·) be a Boolean ring. Define operations , and " ' " on R by

(i) $a \quad b = a + b + ab$

(ii) $a \quad b = a \cdot b$

(iii) $a' = 1 + \alpha$

then prove that (R, , , ' , 0, 1) is a Boolean algebra.

MA/MSc - Mathematics (Final)
Assignments
MEASURE THEORY AND FUNCTIONAL ANALYSIS

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. (a) Prove that the interval (a,) is measurable, for any real a.

   (b) Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets, that is, a sequence with E_{n+1} E_n for each n. Let $m(E_1)$ be finite. Then prove that

2. (a) State and prove bounded covergence theorem.

   (b) Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \leq \int_E |f|$.

   Does the integrability of $|f|$ imply that of f ?

3. (a) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in$ 𝔅 such that B_α B_β for $\alpha < \beta$. Then prove that there is a unique measurable extended real valued function f on X such that f $\leq \alpha$ on β_α and f α on $x \setminus B_\alpha$.

   $0 < \nu(A)$ $(E) < \infty$

   $M\left(\bigcap_{i=1} E_i\right)$ $= \lim_{n\to\infty} m(E_n)$ $\le M \| \varphi \|_p$

   (b) Let be a signed measure on the measurable space (X, 𝔅). Let E be a measurable set such that . Then prove that there is a positive set A contained in E with

4. (a) State and prove Lebesgue decomposition theorem.

   (b) Let (X, 𝔅,) be a finite measure space and g an integrable function such that for some constant for all simple functions φ. Then prove that $g \in L^q$.

5. (a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N / M is defined by $\| X+M \| = \inf \{\| x+m \| : m \in M\}$, then prove that N / M is a normed linear space. Further, if N is a Banach space then prove that N / M is also a Banach space.

   (b) Let M be a linear subspace of a ormed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 then prove that f can be extended to a functional f_0 defined on M_0 such that $\| f_0 \| = \| f \|$.

6. (a) State and prove the closed graph theorem.

   (b) State and prove the uniform boundedness theorem.

7. (a) State and prove Schwarz inequality.

   (b) Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Then prove that there exists a unique vector $y_0 \in M$ such that $\| x - y_0 \| = d$.

8. (a) If T is an operator on a Hilbert space H for which $(Tx, x) = 0$ for all x, then prove that $T = 0$.

   (b) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and N_1N_2 are normal.

MA/MSc - Mathematics (Final)
Assignments
Optional : LATTICE THOERY

Answer any five of the following.
All questions carrying equal marks — 5x4=20 Marks

1. (a) Prove that every non-empty finite partly ordered set can be represented by a diagram.

   (b) Let P be partly ordered set of locally finite length satisfying the Jordan-Dedekind chain condition. If P contains an element u such that inf $\{u, n\}$ exists for all elements n of p, then prove that a dimension function can be defined on P.

2. (a) Let L be a lattice, then prove that

   (i) $a \wedge a = a$ and $a \vee a = a$ for all a b L.

   (ii) $a \wedge b = a \vee b$ if and only if $a = b$ for all $a, b \in L$.

   (b) Prove that every finite subset of a lattice has an infimum and a supremum.

3. (a) Prove that if a lattice satisfies both the maximum and minimum condition (in particular, if it is of finite length) then it is complete.

   (b) Prove that if we affix found elements to a conditionally complete lattice we obtain a complete lattice.

4. (a) Prove that the dual, every sublattice and every homomorphic image of a distributive lattice is likewise a distributive lattice.

   (b) Prove that a lattice is modular if and only if no sub-lattice of it is isomorphic with the pentagon lattice.

5. (a) Prove that complemented elements of a bounded distributive lattice form a sublattice.

   (b) Let $(B, V, n\ ‘\ , 0, 1)$ be a Boolean algebra. Define binary operations + and "**.**" on B by $a + b = (a \cap b')\ a' \quad b$ and $a\ .\ b = a \quad b$ then prove that $(B, +, .)$ is a Boolean ring.

6. (a) Prove that a valuation of a Boolean algebra is additive if and only if $(0) = 0$.

   (b) Verify that in every Boolean algebra $ed(a', b', c') = (\quad ed(a, b, c))'$.

7. (a) Prove that every ideal and dual ideal of a lattice is a convex sub lattice. Also prove that every convex sub lattice of L is the set intersection and an ideal and of a dual ideal.

   (b) Prove that the set union of any ideal chain of a lattice L is itself an ideal in L.

8. (a) Prove that a lattice is distributive if and only if for any two distinct elements of the lattice there exists a prime ideal which includes one of the elements without including the other.

   (b) Prove that if and are permutable equivalence relations of a set then their product is also an equivalence relation of and = p.

MA/MSc - Mathematics (Final)
Assignments

Optional : LINEAR PROGRAMMING AND GAME THEORY

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. (a) Let A be an m x n matrix over **R**. Let b $\in$ $\mathbf{R}^n$. Prove that exactly one of the following alternatives holds.

   (i) The equation $xA = b$ has a solution.

   (ii) The equation $Ay = 10$, $by = 1$ have a solution.

   (b) State and prove duality theorem.

2. (a) State and prove optimality criterion.

   (b) Prove that a transportation problem is feasible if and only if total supply is greater than or equal to total demand.

3. (a) Explain the concept of replacement operation. Using replacement operation, find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b) State and prove equilibrium theorem.

4 (a) Solve the following L.P.P. by the simplex method.

   Maximize $z = 8x_1 + 19x_2 + 7x_3$.

   Subject to

   $3x_1 + 4x_2 + x_3 \leq 25$

   $x_1 + 3x_2 + 3x_3 \leq 50$

   $x_i \quad 0 \ (i = 1,2).$

   (b) Solve :

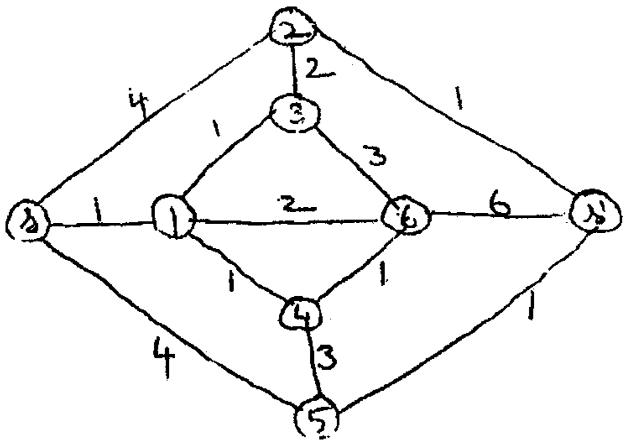
   $3x_1 + x_2 - x_3 \leq 1$

   $-x_1 - x_2 + 3x_3 \leq -4$

   $-x_1 + 2x_2 - 2x_3 \leq -3$

   $x_1 + 2x_2 + 2x_3 \leq -3$

5. State prove max flow - min cut theorem, using this, find the maximal flow from s to s' in the following network.



6. (a) Solve the following assignment problem

	J_1	J_2	J_3	J_4	J_5
I_1	9	6	7	0	5
I_2	5	5	1	1	8
I_3	1	3	5	6	4
I_4	5	2	3	0	0
I_5	12	2	11	10	12

(b) Solve the following transportation problem:

					Suppliers
	3	3	8	2	2
	1	5	4	6	5
	2	4	7	7	7
Demands	2	4	4	4	

$$\begin{pmatrix} 0 & 3 & -7 \\ -1 & -3 & 2 \\ -7 & 0 & -6 \end{pmatrix}$$

7. (a) Explain the following :
   (i) Game
   (ii) Pure Strategies
   (iii) Mixed Strategies
   (iv) Extended game.

   (b) State and prove minimax theorem.

8. (a) State and prove the fundamental theorem of game theory.

   (b) Solve the following game by the linear programming method.

MA/MSc - Mathematics (Final)
Assignments
CUMMUTATIVE ALGEBRA

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. Let $I_1, I_2 ......\ I_n$ be ideals of a ring A. Define the map $: A \to \quad A / I_i$ by

   $(x) = (x+I_1,\ x+I_2,\ .....\ x+I_n)$ then prove that the following hold.

   (a) If I_i and I_j are coprime for $i \neq j$ then

   (b) is a homomorphism.

   (c) is surtective (i.e., onto) if and only if I_i and I_j are coprime for $i \neq j$.

   (d) is injective (i.e., one one) if and only if $= \{0\}$.

2. (a) Let L M N be A-modules. Then show that

   (i) $$\frac{(L/N)}{(M/N)} \cong \frac{L}{M} \qquad \prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$$

   (ii) If m_1, m_2 are sub modules of m, then show that $\frac{m_1+m_2}{m_1} \cong \frac{m_2}{m_1 \cap m_2}$.

   (b) State and prove Nakayama's lemma.

3. (a) Let m and N be A-modules. Then prove that the tensor product of M and N exists.

   (b) If M and N are finitely generated then prove that $M \otimes N$ is finitely generated.

4. (a) Let M, M', M'' be A-modules. Then prove that if $m' \quad m \quad m''$ is exact at m then $s^{-1}m^{-1} \quad s^{-1}m \quad s^{-1}m''$ is exact at $S^{-1}m$.

   (b) (i) Let m be a finitely generated A-module, S be a multiplicatively closed subset of A. Then prove that $s^{-1}Ann(m_1) = Ann(s^{-1}m)$.

   (ii) If N, P are sub modules of an A-module M and P is finitely generated then prove that $S^{-1}(N : P) = (s^{-1}N : S^{-1}P)$.

5. (a) Define the concept of a primary ideal prove that every prime ideal is primary. Is the converse true? Justify.

   (b) State and prove second uniqueness theorem.

6. (a) Define the following concepts in relation to an ideal I of a ring A.

   (i) isolated prime ideal belonging to A.

   (ii) isolated set of prime ideal's belonging to A.

   (b) State and prove going down theorem.

7. (a) Prove that a partially ordered set X satisfies the ascending chain condition if and only if it satisfies the maximal condition.

   (b) State and prove Hilbert basis theorem.

8. (a) In a Northerian ring prove that every ideal has a primary decomposition.

   (b) Prove that the following are equivalent for any Artin local ring A with the following maximal ideal M.

   (i) every ideal of A is principal.

   (ii) the unique maximal ideal M is principal.

   (iii) $\dim_F$ ≤ 1 where F =

$$\left(\frac{A/M}{M/M^{2}}\right)$$

MA/MSc - Mathematics (Final)
Assignments
Optional : INTEGRAL EQUATIONS

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

1. (a) Form the integral equation corresponding to the initial value problem + y = cosx, $y(0)=0$; $y'(0)=1$.

   (b) If φ_1 and φ_2 are two linearly independent solutions of $(p(x)y'(x))' - q(x)y = 0$ where p,q are real valued continuous functions on $[x_1,x_2]$ then show that $p(x)(\varphi_2(x)\varphi_1'(x) - \varphi_1(x)\varphi_1'(x))$ is a constant.

2. (a) Let k = $[a,b]$ x $[a,b]\to$R is continuous and : [a,b]→R is bounded.

   Define $F : [a,b]\to R$ by $F(x) =$ $k(x,y)$ $(y)dy$. Then prove that F is continuous on $[a,b]$.

   (b) Find the solution of

   $u'(x) +$ $exp\{x - y\}u(y)dy = f(x)$, $0 \le x \le 1$ with the initial condition $u(0) = 0$.

$\varphi_j(x)\overline{\phi_j(y)}$

3. (a) Let $f : [a, b]\to R$ be continuous and the Kernel $K(x,y)$ be degenerate on $[a,b]$ x $[a,b]$. Consider the non-homogenous integral equation $\varphi(x) =$ $K(x,y)\varphi(y)dy + f(x)$ with

   $\|K\|_2 <$. Then prove that the successive approximations defined by

   $\varphi(x) = f(x)$ $\varphi_{j+1}(x) =$ $K(x,y)\varphi_j(y)dy + f(x),\ x \in [a,b],\ j \in \{0,1,2,....\}$ converges uniformly to a solution of the given integral equation on $[a,b]$.

   (b) If ^{2} is an eigen value of K_{Left} with eigen function φ then prove that

   $(x) =$ $K(x,y)\varphi(y)\,dy$ is an eigen function of K_{right} with eigen value ^{2}.

4. (a) If $K(x,y)$ is Hermitian Kernel with $_j$'s are eigen values of K and φ_j's are corresponding eigen functions then prove that $K(x,y)$ has a series expansion of the form

   K(x,y) = provided the series on the right converges in the mean, i.e., in $L^2[a,b]$, or uniformly on $[a,b]$ x $[a,b]$.

5. (a) Solve the integral equation $\varphi(x) = \lambda \int_0^x e^{K(x-y)} \varphi(y)dy + f(x)$.

(b) Solve the integral equation $\int_0^x \varphi(x - y) \{\varphi(y) - 2 \sin ay\}dy = x \cos ax$ by using Laplace transforms.

6. (a) Solve the integral equation $\varphi(x) = \int_{-\infty}^{+\infty} e^{iwx} \varphi(w)\, dw$.

(b) Define Hilbert transform and solve the integral equation $\frac{1}{w^2 + x^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(u)}{x-u} du$, a > 0.

7. Write Picard's method to solve nonlinear fredhohm integral equation $\varphi(x) = f(x) + \lambda \int f(x,y,\varphi(y))dy$. Further, find the error estimate in the approximation.

8. (a) Explain the main theme of Gelarkin's method to find approximate solution of Fredholm integral equation of first kind.

(b) If $K(x,y) = \sum_{j=1}^{\infty} \frac{e_j(x)e_j(y)}{\lambda_j}$ where $\{e_j\}$ is a complete orthonormal base for L^2 and λ_n are the eigen values of K and if $K_n(x,y) = \int K(x,z)K_{n-1}(z,y)dz$ then prove that

$\int K_n(x,z)f(z)dz = \sum_{j=1}^{\infty} \frac{f_j e_j(x)}{\lambda_j^n}$ for any $f \in L^2$ with $f_j = (f, e_j)$.

MA/MSc - Mathematics (Final)
Assignments
NUMBER THEORY

Answer any five of the following.
All questions carrying equal marks 5x4=20 Marks

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1. (a) Prove that the Euler-Totient function has the following properties.

   (i) $\varphi(p^{\alpha}) = p^{\alpha} = p^{\alpha-1}$ for all primes p and all $\alpha \geq 1$.

   (ii) $\varphi(mn) = \varphi(m_1.\varphi)(n)$. where d = (m,n) the greatest common divisor of m and n.

   (b) Let f be a multiplicative function. Then prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n).f(n)$ for all $n \geq 1$.

2 (a) If x 1, then prove that $= \log x + C + o$ where C is the Euler constant.

   (b) Prove that the set of lattice points visible from the origin has density .

3. (a) State and prove Abel's Identity

   (b) Prove that, for x 2. $\frac{n}{\log n}$

   (i) $(x) = (x)\log x -$ dt and

   (ii) $(x) =$ - dt.

4. (a) For $n \geq 2$ prove that .

   (b) For any integer a and any prime p, prove that $a^p \equiv a(\text{mod}\, p)$.

5. (a) Prove that any abelian group of order n has exactly n-distinct characters.

   (b) If G is a finite abelian group with elements $a_1, a_2, ...a_n$ and $f_1, f_2, ...f_n$ are characters of G then prove that $= \begin{cases} n \text{ if } a_i = a_j \\ 0 \text{ if } a_i \neq a_j \end{cases}$.

6. (a) Out line the plan of the proof of the Dirichlet theorem.

   (b) Prove that there are infinitely many primes of the form $4n + 1$.

7. (a) Let p be an odd prime. Then prove that every residue system modulo m has exactly Quadratic residues and exactly Quadratic non residues *mod p*.

(b) State and prove quadratic reciprocity law.

8. (a) State and prove Gauss lemma.

(b) Define *S(a;m)* and prove that *S(a,m)* = .

$$\sqrt{\frac{m}{2a}}\left(\frac{1+i}{\sqrt{2}}\right)\overline{S(m,a)}$$