M.A. / M.SC., Mathematics (Final Year)

ASSIGNMENTS

Measure Theory and Functional Analysis

Answer any five of the following. All question carry equal marks.

5 x 4 = 20 Marks

1. (a) Define outer measure and prove that the outer measure of an interval is its length.

(b) Define measurable set and prove that the interval \((a, \infty)\) is measurable for any real number \(a\).

2. (a) State and prove bounded convergence theorem.

(b) Let \(f\) be a non negative function which is integrable over a set \(E\). Then prove that for given \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every set \(A \subseteq E\) with \(m(A) < \delta\) we have

\[
\int_A f < \varepsilon.
\]

3. (a) i) Define \(\sigma\)– algebra.

ii) Let \(B\) be a \(\sigma\)– algebra. If \(E_i \in B\) and \(\mu(E_i) < \infty\) with \(E_i \supset E_{i+1}\) then prove that

\[
\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu(E_n)
\]

(b) State and prove Fatou's lemma

4. (a) i) Define signed measure

ii) State and prove Hahn decomposition theorem.

(b) State and prove Radon - Nikodym theorem.

5. (a) i) Define normed linear space and Banach space.

ii) Let \(M\) be a closed linear subspace of a normed linear space \(N\). If the norm of a coset \(x + M\) in the quotient space \(N/M\) is defined by

\[
\|x + M\| = \inf \{\|x + m\| : m \in M\}
\]

then prove that \(N/M\) is a normed linear space. Further, if \(N\) is a Banach space, then prove that \(N/M\) is also a Banach space.

(b) State and prove the Hahn - Banach theorem.
6. (a) State and prove the open mapping theorem.
(b) State and prove the uniform boundedness theorem.

7. (a) i) Define Hilbert space.
ii) State and prove Schwarz inequality.
(b) Prove that a closed convex subset C of a Hilbert space $H$ contains a unique vector of smallest norm.

8. (a) Let $H$ be a Hilbert space, and let $f$ be an arbitrary functional in $H^*$. Then prove that there exists a unique vector $y$ in $H$ such that $f(x) = (x, y)$ for every $x$ in $H$.
(b) Prove that the adjoint operation $T 	o T^*$ on $B(H)$ has the following properties.

i) $(T_1 + T_2)^* = T_1^* + T_2^*$
ii) $(\alpha T)^* = \overline{\alpha} T^*$
iii) $(T_1 T_2)^* = T_2^* T_1^*$
iv) $T^{**} = T$
v) $\|T^*\| = \|T\|$  
vii) $\|T^* T\| = \|T\|^2$
Answer any five of the following. All question carry equal marks.

1. (a) Let $G$ be an open and connected set in the complex plane and if $f : G \rightarrow \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z$ in $G$ then prove that $f$ is constant.

(b) Let $u$ and $v$ be real-valued functions defined on a region $G$ and suppose that $u$ and $v$ have continuous partial derivatives. Then prove that $f : G \rightarrow \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if $u$ and $v$ satisfy Cauchy - Riemann Equations.

(or)

2. (a) If $G$ is an open set in the Complex Plane, $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then prove that for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that

$$\Gamma(a) = \gamma(a), \quad \Gamma(b) = \gamma(b) \quad \text{and} \quad \left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$  

(b) State and prove Liouville's theorem and deduce fundamental theorem of algebra.

UNIT - II

3. (a) State and prove the first version of Cauchy's integral formula.

(b) State and prove Morera's Theorem.

(or)

4. (a) State and prove Laurent series development theorem.

(b) Evaluate the following integrals:

(i) $\int_{0}^{\infty} \frac{\sin x}{x} \, dx$

(ii) $\int_{0}^{\pi} \frac{d\theta}{a + \cos \theta}, \quad a > 1$.

UNIT - III

5. (a) State and prove Montel's Theorem.

(b) Let $G$ be a region in $\mathbb{C}$. If $M(G)$ is the set of all meromorphic functions on $G$ then prove that $M(G) \cup \{\infty\}$ is a complete metric space.

(or)

6. (a) Let $Re z_n > 0$ for all $n \geq 1$. Then prove that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.
(b) State and prove Weierstrass Factorization theorem.

UNIT - IV

7. (a) Let $K$ be a compact subset of the region $G$. Then prove that there are straight line segments $\gamma_1, \ldots, \gamma_n$ in $G \setminus K$ such that for every function $f$ in $H(G)$,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(w)}{w-z} \, dw$$

for all $z$ in $K$. The line segments form a finite number of closed polygons.

(b) State and prove Schwarz reflection principle.

(or)

8. (a) State and prove Monodromy theorem.

(b) Define Mean Value Property. Show that every Harmonic function has Mean Value Property.

9. (a) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{z^n}{n}$.

(b) Define branch of the logarithm. If $G$ is an open connected set and $f$ is a branch of $\log z$ on $G$ then prove that the totality of branches of $\log z$ are the functions $f(z) + 2\pi ki$, $k \in \mathbb{Z}$.

(c) Define (i) isolated singularity and (ii) removable singularity.

(d) Define meromorphic function and state argument principle.

(e) Let $(S, d)$ be a metric space. Then prove that

$$\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$$

is also a metric on $S$.

(f) Let $G$ be an open subset of the complex plane. Suppose that $H(G)$ is the collection of analytic functions on $G$. If $\{f_n\}$ is a sequence in $H(G)$ and $f$ belongs to $C(G, \mathbb{C})$ such that $f_n \to f$ then prove $f$ is analytic.

(g) Let $V$ and $U$ be open subsets of $\mathbb{C}$ with $V \subset U$ and $\partial V \cap U = \emptyset$. If $H$ is a component of $U$ and $H \cap V \neq \emptyset$ then prove that $H \subset V$.

(h) Define harmonic function. If $u: G \to \mathbb{C}$ is harmonic then prove that $u$ is infinitely differentiable.
Answer any five of the following. All question carry equal marks.

1. (a) For $n \geq 1$ show that $\phi(n) = n \pi \left(1 - 1/p\right)

(b) Show that if both $g$ and $f \ast g$ are multiplicative then $f$ is also multiplicative.

(or)

2. (a) State and prove Euler’s Summation formula.

(b) For all $x \geq 1$ show that

$$\sum_{n \leq x} d(n) = x \log x + (2 \log 2 - 1) x + O\left(\sqrt{x}\right).$$

UNIT - II

3. Show that for every integer

$$\mu \geq 2 \frac{1}{6} \frac{n}{\log n} < \pi(n) < 6 \frac{n}{\log n}$$

(or)

4. (a) State and prove Lagrange’s theorem

(b) Show that for any prime $p$ all the coefficients of the polynomial

$$f(x) = (x-1)(x-2) \ldots (x-p+1) - x^{p-1} + 1$$

are divisible by $p$.

UNIT - III

5. (a) For any real valued character $\chi \mod k$, let $A(n) = \sum_{d|n} \chi(d)$ and $B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}}$ then

Show that

(i) $B(x) \to \infty$ as $x \to \infty$

(ii) $B(x) = 2\sqrt{x} L(1, \chi) + O(1)$ for all $x \geq 1$

(or)

6. (a) For $x > 1$ and $\chi \neq \chi_1$, show that

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O(1)$$

(b) For $x > 1$ and $\chi \neq \chi_1$, show that
\[ L(1, x) \sum_{n \leq x} \frac{\mu(n)x(n)}{n} = o(1) \]

UNIT - IV

7. (a) For every odd prime \( p \) show that

\[ (2/p) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \mod 8 \\ -1 & \text{if } p \equiv \pm 3 \mod 8. \end{cases} \]

(b) State and prove Quadratic reciprocity law.

(or)

8. (a) Determine whether 888 is a quadratic residue or non residue of the prime 1999.

(b) Let \( p \) and \( q \) be distinct odd primes and \( \chi \) be the quadratic character \( \mod p \) then show that the quadratic reciprocity law is equivalent to the congruence

\[ G(1, \chi)^{x^{-1}} = (q/p) \mod q. \]

9. (a) If \( n \geq 1 \) show that \( \sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor \)

(b) For \( x > 1 \) show that \( \sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x) \)

(c) State and prove Abel's identity

(d) State and prove Euler-Fermat theorem

(e) Show that there are infinitely many primes of the form \( 4n-1 \)

(f) Show that there are infinitely many primes of the form \( 4n+1 \)

(g) Let \( p \) be an odd prime. Then show that for all \( n, (n/p) = n^{(p-1)/2} \mod p \)

(h) Determine whether 219 is a quadratic residue or non residue \( \mod 383 \).
Answer any five of the following. All questions carry equal marks.  
5 x 4 = 20 Marks

Section - A

Unit - I

1. (a) Define Lattice (i) as a poset and (ii) as an algebra. Prove that these two definitions are equivalent.

(b) Draw the Hasse diagram of the partially ordered set of all divisors of 36 with respect to the partial ordering divisibility. Determine (i) all atoms if exists, (ii) all dual atoms if exists.

(or)

2. (a) Define complemented lattice. Give an example of a complemented lattice.

(b) Define (i) ideal of a lattice (ii) sublattice. Prove that every ideal is a sublattice. Is converse true? Justify your answer.

Unit - II

3. (a) Prove that a modular lattice L is distributive if and only if no sublattice of L is isomorphic to M₅.

(b) Prove that every element in a lattice satisfying maximum condition is compact.

(or)

4. (a) Define (i) distributive lattice, (ii) modular lattice. Prove that every distributive lattice is modular. Is converse true? Justify your answer.

Unit - III

5. (a) Define a Boolean algebra. State and prove De Morgan formulae in Boolean algebra.

(b) Prove that a complete Boolean algebra in atomic if and only if it is isomorphic to the Boolean algebra of the set of all subsets of some set.

(or)

6. (a) Let \( B = ( B \land V, \lor, 0, 1 ) \) be a Boolean algebra. Define the binary operations \( + \) and \( \cdot \) on \( B \) by

\[
    a + b = ( a \land b^\lor ) \lor ( a^\lor \land b )
\]

\[
    a \cdot b = a \land b
\]

Prove that \( (B, +, \cdot) \) is a Boolean ring.

Unit - IV

7. (a) Prove that every convex sublattice of a lattice \( L \) is the intersection of an ideal and a dual ideal.
(b) Prove that every distributive lattice is isomorphic to a ring of sets (or)

8. (a) Let $\theta$ be a congruence relation on a lattice $L$. Let $a, b \in L$. Prove that the following statements are equivalent

(i) $a \equiv b$ ($\theta$)
(ii) $a \land b \equiv a \lor b$ ($\theta$)
(iii) $x \equiv y$ ($\theta$) for any $x, y \in [a \land b, a \lor b]$

(b) Prove that every ideal of a lattice $L$ is the Kernel of at least one congruence relation if and only if $L$ is distributive

Section - B

9. Answer any FIVE of the following questions. Each question carries 5 marks. ($5 \times 4 = 20$)
(a) Prove that every least element of a poset is minimal. Draw the Hasse diagram of a poset with two minimal elements and no least element
(b) Give example of a lattice with atoms and dual atoms
(c) State and prove fix point theorem in a conditionally complete lattice
(d) In any distributive lattice, prove the following
   
   $a \land x = a \land y, a \lor x = a \lor y \Rightarrow x = y$
(e) Prove that every complete Boolean algebra satisfies infinitely meet distributive law
(f) Prove that a valuation $v$ in a Boolean algebra is additive if and only if $v(0) = 0$
(g) Let $L$ be a lattice satisfying maximum condition. Prove that every ideal of $L$ is principal.
(h) Let $\theta_1, \theta_2, \theta_3$ be congruence relations on $L$ set $M$. If $\theta_1 \leq \theta_3$ and $\theta_1$ is permutable with $\theta_2$, Prove that
   
   $\theta_1 \lor (\theta_2 \land \theta_3) = (\theta_1 \lor \theta_2) \land \theta_3$
UNIT - I

1. i Write FORTRAN expressions corresponding to

(a) \[ \sqrt{x - y^3} - \frac{z^3}{\cos(a + b)} \]
(b) \[ \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-u)^2}{2\sigma^2}} \]
(c) \[ \left(3 + \frac{a}{b}\right)^{m-1} \]

ii Find the final values of A and J.
X = 3.1, Y = 4.6, L = 2, M = -3
J = SQRT (X^Y)
A = J*Y
J = A*M
A = A + ABS(FLOAT(J))

iii Write a FORTRAN program to find the sum of squares of the upper triangular elements of a real square matrix.

(or)

2. i Explain the terms
   a Relational operators.
   b Computer GOTO statement.
   c Arrays and subscribed variables.

ii Write a FORTRAN program to print 100 entries in the Fibonacci sequence 1,1,2,3,5,8,..

UNIT - II

3. a Given the following values of \( f(x) = \log x \), find the approximate value of \( f'(2.0) \), \( f''(2.0) \) using the methods based on linear and quadratic interpolation. Also obtain an upper bound on the error.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.0</td>
<td>0.69315</td>
</tr>
<tr>
<td>1</td>
<td>2.2</td>
<td>0.78846</td>
</tr>
<tr>
<td>2</td>
<td>2.6</td>
<td>0.95551</td>
</tr>
</tbody>
</table>

b Find the approximate value of \[ I = \int_0^1 \frac{dx}{1+x} \] using (i) trapezoidal rule (ii) Simpson's rule
4. a For the method 

\[ f'(x_0) = -\frac{3f(x_0) + 4f(x_1) - f(x_2)}{2h} + \frac{h^2}{3} f''(\xi), \quad x_0 < \xi < x_2 \]

determine the optimal value of h, using the criteria \( |RE| = |TR| \). Using this method and the above value of h, determine approximate value of \( f'(2.0) \) from the following tabulated values of \( f(x) = \log(x) \).

<table>
<thead>
<tr>
<th>x</th>
<th>2.0</th>
<th>2.01</th>
<th>2.02</th>
<th>2.06</th>
<th>2.12</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.69315</td>
<td>0.69813</td>
<td>0.70310</td>
<td>0.72271</td>
<td>0.75142</td>
</tr>
</tbody>
</table>

b. Evaluate the integral \( I = \int_1^4 \frac{dx}{x^2 + 1} \) using Gauss- Legendre three point formula

**UNIT - III**

5. a Solve the initial value problem \( u' = -2 + u^2, u(0) = 1 \) with h = 0.2 on the interval [0,1] using the fourth order Classical Runge-Kutta Method.

b Solve the initial value problem \( u' = -2 + u^2, u(0) = 1 \) with h = 0.2 interval [0,0.4] using Predictor-Corrector Method.

(or)

6. a Solve the system of equations \( u' = -3u + 2v, v' = 3u - 4v, u(0) = 1, v(0) = 0.5, \) using Euler Cauchy method on the interval [0,1] with h = 0.2.

b. Use the Numerov method to solve the boundary value problem 
\( u'' = u(0) = 0, u(1) = 0 \) with h = 1/4.

**UNIT - IV**

7. Write FORTRAN program to evaluate \( \int \frac{dx}{1 + e^x} \) using composite trapezoidal rule.

(or)

8. Write a FORTRAN program to obtain numerical solution for the initial value problem \( u' = -2 + u^2, u(0) = 1 \) on [0,10], using Euler's method with h = 0.2.

Answer any FOUR of the following questions. Each question carries 5 marks. \( 4 \times 5 = 20 \)

9. a Write the rules relating to nesting of DO loops

b Explain logical IF statement

c Explain relational operators

d Explain Radon method of Numerical Integration.

e Explain Predictor-Corrector method.

f Define a step size h that has to be used in the tabulation of linear interpolation that will be correct up to four decimal places.

g Write a FORTRAN program to find the maximum of a given list of numbers.

h Calculate the \( n^{th} \) divided difference of \( f(x) = 1/x \).
SCHOOL OF DISTANCE EDUCATION
M.A. / M.SC., Mathematics (Final Year)

ASSIGNMENTS
(Optional) : LINEAR PROGRAMMING AND GAME THEORY

Answer any five of the following. All questions carry equal marks.

5 x 4 = 20 Marks

1. (a) Define
   (i) Linear Programming Problem (L.P.P.)
   (ii) Solution of an L.P.P.,
   (iii) Feasible solution to an L.P.P.
   (iv) Optimal solution to an L.P.P.

   (b) State and prove the optimality criterion for the standard maximum problem.

2. (a) Let \( A \) be an \( m \times n \) matrix and \( b \in \mathbb{R}^n \). Prove that exactly one of the following alternatives hold.

   (i) the equation \( Ax = b \) has a non negative solution.

   (ii) the inequalities \( Ay \geq 0 \) and \( by < 0 \) have a solution

   (b) Define a convex cone. For any \( b \in \mathbb{R}^n \), prove that the set \( \{ \lambda b / \lambda \geq 0 \} \) is a convex cone.

UNIT-II

3. (a) State and prove the fundamental duality theorem

   (b) Solve the following canonical maximum problem by computing all basic solutions.

   \[
   4\xi_1 + 2\xi_2 + \xi_3 = 4 \\
   \xi_1 + 3\xi_2 = 5 \\
   2\xi_1 + 3\xi_2 = \text{maximum}
   \]

4. (a) Invert the following matrix using replacement operation

   \[
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & -1 & 1 \\
   2 & 1 & -1 \\
   \end{bmatrix}
   \]

   (b) Solve the following L.P.P. by the simple method

   Maximize \( 2\xi_1 + 4\xi_2 + \xi_3 + \xi_4 \)

   Subject to

   \[
   \begin{align*}
   \xi_1 + 3\xi_2 + \xi_4 & \leq 4 \\
   2\xi_1 + \xi_2 & \leq 3 \\
   \xi_i & \geq 0 (i = 1, 2, 3, 4)
   \end{align*}
   \]
UNIT - III

5. (a) State and prove max flow min cut theorem
(b) Solve the transhipment problem

<table>
<thead>
<tr>
<th></th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
<th>$\sigma_i$</th>
</tr>
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<td>7</td>
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<tr>
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<td></td>
</tr>
</tbody>
</table>

6. (a) Solve the following assignment problem

<table>
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<td>11</td>
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<td>10</td>
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</table>

(b) Solve the following transportation problem.

<table>
<thead>
<tr>
<th>Plants</th>
<th>$P_1$</th>
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<th>4</th>
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<tr>
<td>$P_3$</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

UNIT - IV

7. (a) State and prove minimax theorem. Using this theorem solve the game with pay off matrix.

$$
\begin{bmatrix}
-1 & 6 & 5 \\
-2 & -1 & 4 \\
-3 & 0 & -2
\end{bmatrix}
$$

(b) State and prove the fundamental theorem of game theory.

8. (a) What is the relation between matrix game and linear programming.
(b) Solve the following game by linear programming method.

$$
\begin{bmatrix}
3 & -4 & 2 \\
1 & -3 & -7
\end{bmatrix}
$$
9. (a) Explain the transportation problem
(b) Prove that the dual of the dual is primal
(c) Write the dual of the L.P.P.
   Minimize $3x_1 + 4x_2 - 3x_3$
   Subject to
   
   \[
   \begin{align*}
   x_1 + x_2 - x_3 &= 4 \\
   x_1 - x_2 &\geq 2 \\
   x_1 - x_3 &\geq 3 \\
   x_2 &\text{ is unrestricted, } x_1 \geq 0, x_3 \geq 0.
   \end{align*}
   \]

(d) Solve the following by replacement operation.
   
   \[
   \begin{align*}
   x_1 - x_2 &= 1 \\
   2x_1 - 3x_2 &= 1
   \end{align*}
   \]

(e) If $f$ is a flow from $s$ to $s'$ in the network $(N, k)$, prove that $f(s, N) = f(N, s')$.

(f) State feasibility theorem for transshipment problem

(g) Prove that each game has at most one value

(h) Explain the symmetrization of a given game.
Section - A
Unit - I

1. (a) Define nilradical of a ring and prove that the nilradical of A is the intersection of all the prime ideals of A.
(b) Show that

(i) \( r(1) \geq 1 \)
(ii) \( r(r(1)) = r(1) \)
(iii) \( r(1I) = r(1) \cap r(I) = r(1) \cap r(J) \)
(iv) \( r(I) = (1) \Leftrightarrow I = (1) \)

(or)

2. (a) (i) If \( L \supseteq M \supseteq N \) are A-modules, then show that \( (L/N)/(M/N) \cong L/M \).

(ii) If \( M_1, M_2 \) are submodules of \( M \), then show that \( (M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2) \).
(b) State and prove Nakayama’s lemma.

Unit - II

3. (a) Let \( M \) be an A-Module, \( P \) a B-Module and \( N \) an \( (A,B) \)-bimodule. Show that \( M \otimes_A N \) is a B-Module and \( N \otimes_B P \) is an A-Module and \( (M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P) \).
(b) For an A-Module \( N \), show that the following are equivalent
   i) \( N \) is flat.
   ii) If \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) is any exact sequence of A-Modules, then tensored sequence \( 0 \rightarrow M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0 \) is exact.
   iii) If \( f : M' \rightarrow M \) is injective, then \( f \otimes 1 : M' \otimes N \rightarrow M \otimes N \) is injective.
   iv) If \( f : M' \rightarrow M \) is injective, and \( M, M' \) are finitely generated, then \( f \otimes 1 : M' \otimes N \rightarrow M \otimes N \) is injective.

(or)

4. (a) For any A-module \( M \), show that the following are equivalent
   i) \( M \) is flat A-Module
   ii) \( M_p \) is a flat \( A_p \)-Module for each Prime ideal \( p \) of \( A \)
   iii) \( M_m \) is flat \( A_m \) Module for each maximal ideal \( m \) of \( A \).
(b) Let \( M \) be a Finitely generated A-module, \( S \) a multiplicatively closed subset of \( A \). Then
show that.
\[ S^{-1}(Ann(M)) = Ann(S^{-1}M) \]

**Unit - III**

5. (a) Let \( Q \) be a primary ideal of \( A \), \( r(Q) = p \) and \( x \in A \). prove that
   i) \( x \in Q \) then \( (Q : x) = A \)
   ii) \( x \notin Q \) then \( (Q : x) \) is \( P \)-primary
   iii) \( x \notin P \) then \( (Q : x) = Q \).
(b) state and prove. 1st Uniqueness theorem.
   (or)

6. (a) Let \( A \subseteq B \) be integral domains, \( B \) integral over \( A \). Then prove that \( B \) is a field if and only if \( A \) is a field.
   b) State and prove Going-down theorem.

**Unit - IV**

7. (a) Let \( A \subseteq B \) be integral domains, \( B \) finitely generated over \( A \). Let \( V \) be a non-zero element of \( B \). Then there exists \( u \neq 0 \) in \( A \) with the following property: any homomorphism \( f \) of \( A \) into an algebraically closed field \( \Omega \) \( f(u) \neq 0 \) can be extended to a homomorphism \( g \) of \( B \) into \( \Omega \) such that \( g(V) \neq 0 \)
(b) \( M \) is a Noetherian \( A \)-module if and only if every submodule of \( M \) is finitely generated.
   (or)

8. (a) State and prove Hilbert's Basis theorem.
   (b) Let \( A \) be an artin local ring, then show that the following are equivalent
      i) Every ideal in \( A \) is principal
      ii) the Maximal ideal in \( A \) is principal
      iii) \( \dim_{K}(m/m^{2}) \leq 1 \)

**Section - B**

9. Answer any FIVE of the following questions. Each question carries 5 marks. \( 5 \times 4 = 20 \)
   (a) If \( x \) is a nilpotent element of \( A \), show that \( 1+x \) is a unit of \( A \)
   (b) Let \( M \) be an \( A \)-module and \( N, P \) are submodules of \( M \), Then show that \( (N: P) = \text{Ann}(N+P)/ \text{Ann}(N) \).
   (c) Let \( A \) be a local ring, \( M \) and \( N \) finitely generated \( A \)-modules, such that \( M \otimes_{A} N = 0 \). Then prove that \( M=0 \) or \( N=0 \)
   (d) Let \( A \rightarrow B \) be a ring homomorphism and let \( p \) be a prime ideal of \( A \). Then show that \( p \) is the contraction of a prime ideal of \( B \) if and only if \( p = p^c \)
   (e) Let \( S \) be a multiplicatively closed subset of \( A \), and \( q \) be a \( p \)-primary ideal. If \( S \cap p = \emptyset \), then show that \( S^{-1}q = S^{-1}A \)
   (f) Let \( A \subseteq B \) be rings, \( B \) integral over \( A \).
      i) If \( b \) is an ideal of \( B \) and \( a = b \cap A \cap A \), then show that \( B/b \) is integral over \( A/a \)
      ii) If \( S \) is a Multiplicatively closed subset of \( A \), then show that \( S^{-1}B \) is integral over \( S^{-1}A \)
   (g) Show that the length \( l(M) \) is an additive function on the class of all \( A \)-modules of finite length.
   (h) Show that in a noetherian ring every irreducible ideal is primary.
M.A. / M.SC., Mathematics (Final Year)

ASSIGNMENTS
Integral Equations

Answer any five of the following. All question carry equal marks.

5x4=20 Marks

1. (a) Suppose an elastic string is fixed at two distinct end points. If the weight is attached between these points, find the integral representation for the displacement of the string due to the complete weight distribution, assuming that the tension in the string is uniform.

(b) Find the eigen values \( \lambda_n \) and eigen functions \( \phi_n \) for \( y'' + \lambda y = 0 \) with the boundary condition \( y(0) = 0, y(a) = 0 \).

2. (a) Transform the differential equation.

\[
y'' + 2xy' + y = 0 \quad \text{satisfying} \quad y(0) = 1, \quad y'(0) = 0
\]

into an integral equation.

(b) Find the integral equation for the problem defined by

\[
\frac{d^2y}{dx^2} + 4y = f(x), \quad 0 \leq x \leq \frac{\pi}{2} \quad \text{satisfying} \quad y(0) = 0 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = 0.
\]

3. (a) Solve.

\[
3x^2 + 4x = \int_1^x (6x^2 + 4xy^2) \phi(y) \, dy
\]

(b) Find the eigen values and eigen functions of the system

\[
\phi(x) = \lambda \int_0^1 (1 + xt) \phi(t) dt, \quad 0 \leq x \leq 1
\]

4. (a) If \( \phi(x) = \lambda \int_0^1 k(x, y) \phi(y) \, dy \), where \( k \) is a Hermitian kernel, has eigen values then show that they are all seal.

(b) If \( k \) is Hermitian then prove that the eigen functions corresponding to different eigen values are orthogonal.

5. (a) Solve the integral equation

\[
\phi(x) = 3 \int_0^x \cos(x - y) \phi(y) \, dy + e^x \quad \text{with} \quad \phi(0) = 1.
\]

(b) Solve the integral equation
\[ \varphi(x) = \lambda \int_0^x e^{x-y} \varphi(y) \, dy + f(x) \] by using the resolvent kernel method.

6. (a) Solve the integral equation

\[ \varphi(x) = x^3 + \int_0^x e^{3x-y} \varphi(y) \, dy \] by using the method of laplace transforms.

(b) Solve the integral equation

\[ \varphi(x) = \int_0^1 \frac{1 + \varphi(y)}{1+y} \, dy \] by using picard's method.

7. (a) Write picard's method for the existence, and hence to find solution of non linear volterra equation of second kind \( \varphi(x) = f(x) + \lambda \int_0^x F(x,y,\varphi(y)) \, dy \).

(b) Find first and second approximation in the iterative solution of the integral equation.

\[ \int_0^1 (x+y)^\frac{1}{2} \left[ \varphi(y) \right]^\frac{1}{2} \, dy = \varphi(x) \] and find bounds on \( \varphi(x) \).

8. (a) Find first three iterates of the solution of \( \varphi(x) = \lambda \int_0^x \sin(xy) \varphi(y) \, dy + 1 \)

(b) Show that the solution of \( \varphi(x) - \int_0^1 e^{xy} \varphi(y) \, dy = 1 - x^{-1}(e^x - 1) \) is \( \varphi(x) = 1 \).

Also, find approximation to \( \varphi\left(\frac{1}{4}\right) \) and \( \varphi\left(\frac{3}{4}\right) \) when \( \varphi(x) \) is determined by the integral equation \( \varphi(x) - \int_0^1 e^{xy} \varphi(y) \, dy = 1 - x^{-1}(e^x - 1) \).
Section - A

1. a) Prove that every distributive Lattice is a modular Lattice.

b) Prove that every complete is isomorphic to the lattice of closed subsets of some set A with a closure operator C.

c) Prove that < con A, ⊆ > is a complete sublattice of the lattice of equivalence relations < Eq (A), ⊆ > on A.

d) Let \( \alpha : A \rightarrow B \) be a homomorphism. Then prove that ker \( \alpha \) is a congruence on A.

e) Prove that every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

f) Let K be a finite set of finite algebras. Then prove that V(k) is a locally finite variety.

g) If R is a Boolean ring then prove that R satisfies the following

\[ x + x = o \text{ and } x.y \approx y.x. \]

h) For \( a/\cup, b/\cup \) in an ultra product \( \Pi A/\cup \), prove that \( a/\cup = b/\cup \) if and only if and only if \( [a = b] \in \cup \).

UNIT - I

2. a) Prove that two lattices \( L_1 \) and \( L_2 \) are isomorphic if and only if there is a bijection \( \alpha \) from \( L_1 \) to \( L_2 \) such that both \( \alpha \) and \( \alpha^{-1} \) are order-preserving.

b) Prove that L is a non-distributive lattice if and only if \( M_5 \) or \( N_5 \) can be embedded into L.

3. a) Let \( \theta \) be the set of all equivalence relations on A and \( a, b \in A \) then prove the following.

\[ A = \bigcup_{a \in A} a/\theta \]

VIII
(ii) \( a/\theta \neq b/\theta \) implies \( a/\theta \cap b/\theta = \phi \).

Where \( a/\theta \) is the equivalence class of \( a \) modulo \( \theta \).

\( b/\theta \) is the equivalence class of \( b \) modulo \( \theta \).

b) If \( C \) is an algebraic closure operator on a set \( A \) then prove that \( L_c \) is an algebraic - lattice, and the compact elements of \( L_c \) are precisely the closed sets \( C(X) \), where \( X \) is a finits subset of \( A \).

**UNIT - II**

4. a) Let \( A \) be an algebra and \( X \subseteq A \) then prove that the subuniverse generated by \( x \) i.e, \( Sg(x) \) is an algebraic closure operator on \( A \).

b) State and prove Tarski's theorem.

**OR**

5. a) If \( A \) is congruence - permutable, then prove that \( A \) is congruence - modular.

b) If \( \phi, \theta \in \text{con} \ A \) and \( \theta \subseteq \phi \) and the map

\[
\alpha : (A/\theta /\phi /\theta ) \to A/\phi \text{ defined by }
\]

\[
\alpha ((a/\theta )/(\phi /\theta )) = a/\phi
\]

Then prove that \( \alpha \) is an isomorphism from \( (A/\theta /\phi /\theta ) \) to \( A/\phi \).

**UNIT - III**

6. a) If \( \theta , \theta ^* \) is a pair of factor congruencies of \( A \), then prove that \( A \cong A/\theta \times A/\theta ^* \) under the map \( \alpha (a) = \{a/\theta , a/\theta ^*\} \).

b) For an indexed family of maps \( \alpha _i : A \to A_i, i \in I \), prove that the following are equivalent :

i) The maps \( \alpha _i \) separate points.

ii) \( \alpha \) is injective, where \( \alpha : A \to \prod _{i \in I} A_i \) is the natural map and is defined by \( (\alpha .a)(i) = \alpha _ia. \)

iii) \( \bigcap _{i \in I} \ker \alpha _i = \Delta \).
7. a) Prove that every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras.

b) Let K be the class of algebras and T(x) be the term algebra where x is a non empty set of variables suppose the term algebra T(x) is exists. Then prove that \( F_k(\bar{x}) \) has the universal mapping property for k over \( \bar{x} \).

**UNIT - IV**

8. Let B be a Boolean algebra. If \( \theta \) is a binary relation on B, then prove that \( \theta \) is a congruence on B if and only if \( \mathcal{F}_\theta \) is an ideal and also prove that for \( a,b \in B, <a,b> \in \theta \) if and only if \( a+b \notin \mathcal{F}_\theta \).

9. a) Let F be a filter [I be an ideal] of B. Then prove that F is an ultra filter [I is a maximal ideal] of B if and only if for any \( a \in B \), exactly one of \( a, a^1 \in F \) [belongs to I].

b) Let B be a Boolean algebra and A any algebra. Prove that \( x = B^* \), a subset S of \( A^x \) is \( A[B]^* \) if and only if S satisfies
   i) the constant functions of \( A^x \) are in S,
   ii) for \( c_1, c_2 \in s, [c_1 = c_2] \) is a clopen subset of x, and
   iii) for \( c_1, c_2 \in s \), and \( N \) a clopen subset of \( x \), \( c_1 \cap N \cup c_2 \cap x \in s. \)